periodica polytechnica

*Mechanical Engineering*
*51/2 (2007) 99–103*
*doi: 10.3311/pp.me.2007-2.09*
*web: http://www.pp.bme.hu/me*


RESEARCH ARTICLE

# Stability of towed wheels with elastic steering mechanism and shimmy damper

*Dénes* Takács / *Gábor* Stépán

Received 2008-01-18

***Abstract***

*This paper investigates a low degree-of-freedom wheel model, which describes the lateral vibration of towed wheels, called the shimmy. The model takes into account the lateral deformation of the tyre and also the torsional elasticity and the damping of the steering mechanism. The lateral deformation of the wheel is modeled by the stretched string-like tyre model, which considers the relaxation of the tyre during rolling. The linear stability analysis of this shimmy model is presented and the stability properties are examined in different parameter ranges of the model.*

***Keywords***

*wheel shimmy · stability*

***Acknowledgement***

*The authors greatly acknowledge the financial support of the Hungarian National Science Foundation under grant no. K 68910, and the Hungarian Academy of Sciences, Research Group on Dynamics of Machines and Vehicles.*

**Dénes Takács**

Department of Applied Mechanics, BME, H-1111 Budapest, Műegyetem rkp. 5., Hungary

e-mail: takacs@mm.bme.hu

**Gábor Stépán**

Department of Applied Mechanics, BME, H-1111 Budapest, Műegyetem rkp. 5., Hungary

e-mail: stepan@mm.bme.hu

## 1 Introduction

The lateral vibration of towed wheels, called the shimmy, is a well known phenomenon in vehicle systems. In spite this motion was already discussed in the early 30's of the last century, there is still no perfect solution to eliminate this dangerous vibration. There are several papers in the literature that study shimmy. Most of the authors consider the elasticity of the suspension system (see [1, 2]) or the elasticity of the wheel [3–5]. Because the majority of vehicles have pneumatic tyres, the second case is more relevant in vehicle system dynamics. Shimmy can occur on motorbikes, airplane nose gears, but the lateral vibrations of trucks, caravans, articulated buses are often called by the name shimmy, too. Of course, the analyses of these three latter cases require a more complex study of the vehicle system. At the investigation of motorbikes and airplane nose gears shimmy, we can obtain fundamental results with a simple towed elastic tyre model [5]. But it is important to note, that the wheels of these vehicles are supported by steering mechanisms. For example on a motorcycle, the rider tries to hold the wheel in the required direction. Additionally, some of the motorcycles have shimmy dampers at the king pin.

This paper investigates a low degree-of-freedom towed wheel model, in which a torsional spring and damper are also considered at the king pin. Similar shimmy models without the above described torsional support were studied in preliminary papers [5, 7]. The most relevant theoretical results were confirmed by experiments, too. In this paper, the stability behaviour of the extended model is investigated, and dimensionless stability charts are presented in different parameter ranges of the model.

## 2 Mechanical model

The tyre model is based on the one given in [5, 6]. As shown in Fig. 1, the elastic wheel is towed by the caster of length $l$ on the steady horizontal ground with constant velocity $v$. The absolute coordinate system $(X, Y, Z)$ is fixed to the ground. During rolling, the tyre adheres to the ground in a contact area. This area is modelled as a contact line of length $2a$. In this way, the deformation of the tyre in the contact patch can be modelled by the lateral displacement $q\,(x, t)$ of this contact line relative to

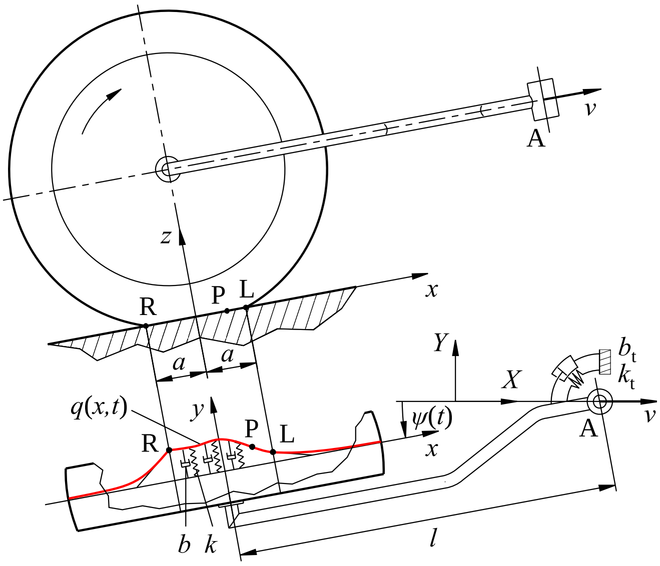



**Fig. 1.** Model of the towed elastic wheel with torsional support

the plane of the wheel. The coordinate $x$ describes the position of the contact points in the contact line, and $t$ stands for the time. Outside the contact patch, the deformation of the tyre can be approximated by decay functions. This technique is well-known in tyre dynamics (see [3, 6]). The deformation function is given by:

$$q(x,t) = \begin{cases} q(-a,t)e^{(x+a)/\sigma}, & \text{if} \quad x \in (-\infty, -a], \\ q(x,t), & \text{if} \quad x \in (-a, a), \\ q(a,t)e^{-(x-a)/\sigma}, & \text{if} \quad x \in [a, \infty), \end{cases} \tag{1}$$

where $\sigma$ is the so-called relaxation length that is an important technical parameter of the tyre.

The caster is supported by a torsional spring and a shimmy damper at the king pin. The torsional stiffness and the torsional damping factor are $k_{\mathrm{t}}$ and $b_{\mathrm{t}}$, respectively.

### 3 Equation of motion

Applying Newton's second law, the equation of motion is given by an integro-differential equation (IDE):

$$J_{\mathrm{A}}\ddot{\psi}(t) = -k\int_{-\infty}^{\infty}(l-x)q(x,t)\mathrm{d}x$$

$$-b\int_{-\infty}^{\infty}(l-x)\dot{q}(x,t)\mathrm{d}x - k_t\psi(t) - b_t\,\dot{\psi}(t), \tag{2}$$

where $J_A$ $\left[\mathrm{kgm}^2\right]$ is the mass moment of inertia of the overall system with respect to the $z$ axis at the king pin, $k$ $\left[\mathrm{N/m}^2\right]$ and $b$ $\left[\mathrm{Ns/m}^2\right]$ are the lateral (specific) stiffness and damping of the tyre distributed along the length, respectively.

The elastic tyre is modeled as a continuum supported by continuously distributed lateral springs. Consequently, the lateral deformation along the contact line is described by a kinematic constraint – namely, we are considering rolling –, which can be given by a partial differential equation (PDE):

$$\dot{q}(x,t) = v\sin\psi(t) + (l-x)\dot{\psi}(t)$$
$$+q'(x,t)\cdot(v\cos\psi(t) - q(x,t)\dot{\psi}(t)) \tag{3}$$

with the boundary condition:

$$q'(a,t) = -\frac{q(a,t)}{\sigma}, \tag{4}$$

where $x \in [-a, a]$ and $t \in [t_0, \infty)$. Dot refers to time derivative, prime refers to differentiation with respect to the spatial coordinate $x$. The PDE specifies that the points of the tyre contacted to the ground have zero speed relative to the ground. The boundary condition comes from the stretched-string tyre theory and gives a simple interpretation of the relaxation length $\sigma$.

The analysis of the system can be carried out by different methods. In [5], the travelling wave-like solution of the linear PDE was calculated and the IDE-PDE system was transformed into a delay differential equation (DDE) with continuous time delay. The time delay describes the memory effect of the contact patch and helps to understand the dynamics of tyres. In this paper, another method is presented: the direct solution of the linear IDE-PDE coupled system leads to the linear stability charts.

### 4 Characteristic equation

If we consider small vibrations, the system can be linearized at the stationary rolling characterized by $\psi \equiv 0$. Using (1), the integrals in (2) can be separated into three ranges. In this way, the linear system is governed by:

$$\ddot{\psi}(t) = -\frac{k}{J_\mathrm{A}}\left(\int_{-\infty}^{-a}(l-x)q(-a,t)e^{\frac{x+a}{\sigma}}\mathrm{d}x + \int_{-a}^{a}(l-x)q(x,t)dx + \int_{a}^{\infty}(l-x)q(a,t)e^{-\frac{x-a}{\sigma}}\mathrm{d}x\right) - \frac{b}{J_\mathrm{A}}\left(\int_{-\infty}^{-a}(l-x)\dot{q}(-a,t)e^{\frac{x+a}{\sigma}}\mathrm{d}x + \int_{-a}^{a}(l-x)\dot{q}(x,t)\mathrm{d}x + \int_{a}^{\infty}(l-x)\dot{q}(a,t)e^{-\frac{x-a}{\sigma}}dx\right) - \frac{k_\mathrm{t}}{J_\mathrm{A}}\psi(t) - \frac{b_\mathrm{t}}{J_\mathrm{A}}\dot{\psi}(t)\ , \tag{5}$$

$$\dot{q}(x,t) = v\ \psi(t) + (l-x)\dot{\psi}(t) + v\,q'(x,t), \tag{6}$$

where the boundary condition $q'(a,t) = -q(a,t)/\sigma,\ x \in [-a,a]$ and $t \in [t_0,\infty)$ still applies.

If we use the standard exponential trial solution

$$\psi(t) = P\,e^{\lambda t} \quad \text{and} \quad q(x,t) = Q(x)\,e^{\lambda t} \tag{7}$$

in (6), we obtain a non-homogeneous linear ordinary differential equation (ODE) with respect to $Q(x)$:

$$Q'(x) - \frac{\lambda}{v}Q(x) = P\frac{\lambda}{v}(x-l) - P \tag{8}$$

that also includes the unknown scalar parameter $P$. The solution of this linear ODE can be calculated as the sum of the homogeneous and the particular solutions:

$$Q(x) = Q_\mathrm{H}(x) + Q_\mathrm{P}(x), \tag{9}$$

where the homogenous solution is the standard exponential solution $Q_\mathrm{H}(x) = H\,e^{\mu x}$ with the characteristic exponent $\mu$. The particular solution can be determined in a polynomial form $Q_\mathrm{P}(x) = a\,x + b$. After the substitution and calculations, the solution of (7) is given by

$$Q(x) = H\,e^{(\lambda/v)x} + P(l-x), \tag{10}$$

where $H$ and $P$ are determined by the initial and the boundary conditions.

If we substitute (7) and (10) into (5) and also into the boundary condition (4), we get a linear system of equations with respect to $H$ and $P$:

$$\begin{bmatrix} a_{11} & a_{12} \\ \left(\frac{\lambda}{v}+\frac{1}{\sigma}\right)e^{\frac{\lambda}{v}a} & \frac{l-a}{\sigma}-1 \end{bmatrix}\begin{bmatrix} H \\ P \end{bmatrix} = \mathbf{0}, \tag{11}$$

where

$$a_{11} = \left(\frac{k+b\lambda}{J_\mathrm{A}}\right)\left(\int_{-\infty}^{-a} e^{-\frac{\lambda}{v}a}e^{\frac{x+a}{\sigma}}(l-x)\mathrm{d}x + \int_{-a}^{a} e^{\frac{\lambda}{v}x}(l-x)\mathrm{d}x + \int_{a}^{\infty} e^{\frac{\lambda}{v}a}e^{-\frac{x-a}{\sigma}}(l-x)\mathrm{d}x\right) \tag{12}$$

and

$$a_{12} = \lambda^2 + \left(\frac{k_\mathrm{t}+b_\mathrm{t}\lambda}{J_\mathrm{A}}\right) + \left(\frac{k+b\lambda}{J_\mathrm{A}}\right)\left(\int_{-\infty}^{-a}(l+a)e^{\frac{x+a}{\sigma}}(l-x)\mathrm{d}x + \int_{-a}^{a}(l-x)^2\mathrm{d}x + \int_{a}^{\infty}(l-a)e^{-\frac{x-a}{\sigma}}(l-x)\mathrm{d}x\right). \tag{13}$$

Because this matrix equation must have non-trivial solutions for $H$ and $P$, the determinant of the coefficient matrix has to be zero. This leads to the transcendental characteristic equation with respect to $\lambda$. It is worth scaling the system into dimensionless form. The introduced parameters are the dimensionless towing length, the dimensionless towing speed, the dimensionless tyre relaxation, the dimensionless torsional stiffness and damping, respectively:

$$L := \frac{l}{a}, \quad V := \frac{v}{2a\,\omega_\mathrm{n}}, \quad \Sigma := \frac{\sigma}{a}, \quad K := \frac{k_\mathrm{t}}{2a^3k}, \quad B := \frac{b_\mathrm{t}}{2a^3b}, \tag{14}$$

where

$$\omega_\mathrm{n} = \sqrt{\frac{2k}{J_\mathrm{A}}\left(a\left(l^2+\frac{a^2}{3}\right) + \sigma\left(l^2+a^2+a\sigma\right)\right) + \frac{k_\mathrm{t}}{J_\mathrm{A}}} \tag{15}$$

is the undamped natural angular frequency of the steady wheel. The damping ratio of the steady wheel is:

$$\zeta = \frac{1}{2\omega_\mathrm{n}}\left(\frac{2b}{J_\mathrm{A}}\left(a\left(l^2+\frac{a^2}{3}\right) + \sigma\left(l^2+a^2+a\sigma\right)\right) + \frac{b_\mathrm{t}}{J_\mathrm{A}}\right). \tag{16}$$

We define also the new dimensionless characteristic exponent:

$$\tilde{\lambda} = \frac{2a}{v}\lambda. \tag{17}$$

With the six dimensionless parameters, the characteristic equation is given by

$$\begin{aligned} D(\tilde{\lambda}) = {} & \Sigma V^2\tilde{\lambda}^3 + 2V(V+\Sigma\zeta)\tilde{\lambda}^2 + (\Sigma+4\zeta V)\tilde{\lambda} + 2 \\ & -2\left(\frac{L-1-\Sigma}{L^2+1/3+\Sigma(L^2+1+\Sigma)+K} + \frac{2\zeta V(L-1-\Sigma)\tilde{\lambda}}{L^2+1/3+\Sigma(L^2+1+\Sigma)+B}\right)\times \\ & \times\left(L\frac{1-e^{-\tilde{\lambda}}}{\tilde{\lambda}} - \frac{1+e^{-\tilde{\lambda}}}{\tilde{\lambda}} + 2\frac{1-e^{-\tilde{\lambda}}}{\tilde{\lambda}^2}\right. \\ & \left. +\Sigma\left(L\frac{1+e^{-\tilde{\lambda}}}{2} - \frac{1-e^{-\tilde{\lambda}}}{2} - \Sigma\frac{1-e^{-\tilde{\lambda}}}{2}\right)\right). \end{aligned} \tag{18}$$

The stationary rolling of the system is exponentially stable if and only if all the infinitely many characteristic exponents have negative real parts.

## 5 Stability analysis

In [7], the stability chart of the undamped system with brush-type tyre model was calculated analytically. It was proved that the undamped system has infinitely many unstable domains in the parameter plane of the towing speed and caster length. It was also confirmed that the number of unstable regions ('lenses') disappear as the damping is increased and only one stability boundary persists. The stability analysis of the mechanical

**Fig. 2.** Stability boundaries of the linear system for different values of K

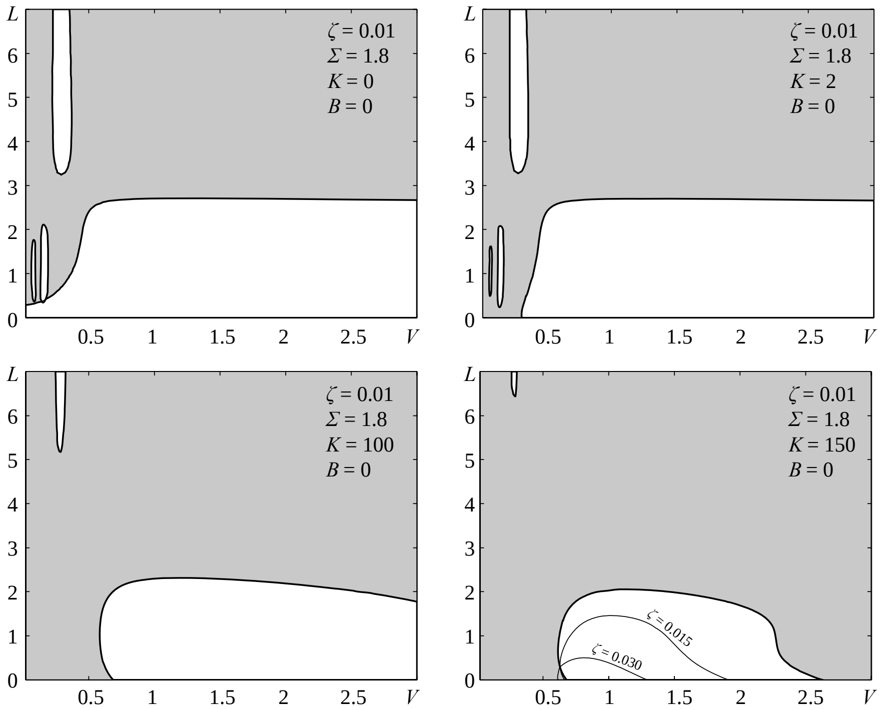
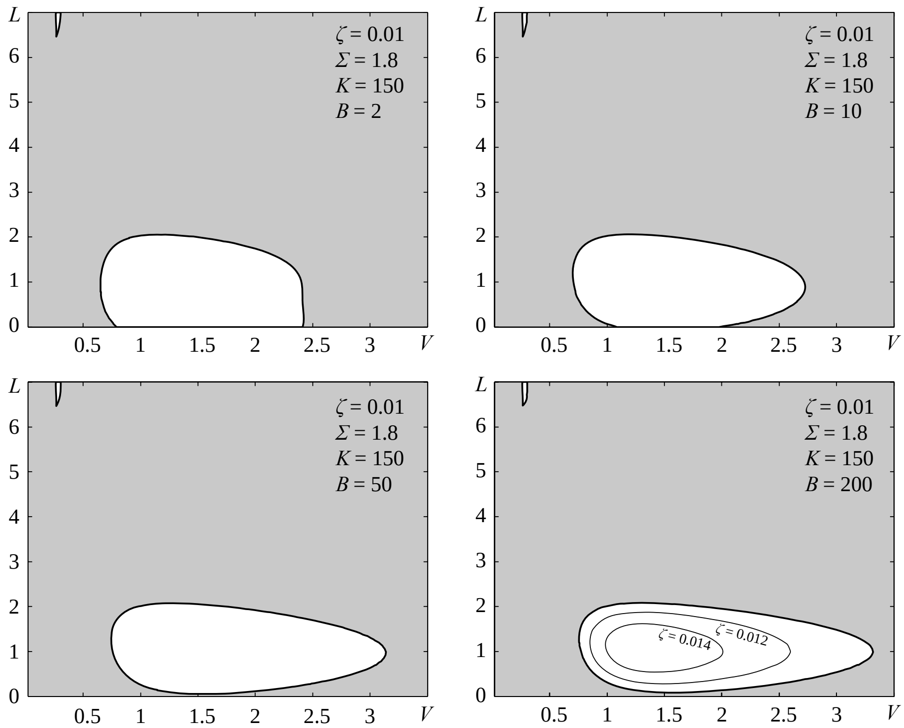



**Fig. 3.** Stability boundaries of the linear system for different values of B




model with stretched string-like tyre model but without torsional stiffness and damping can be found in [5], where the effect of tyre relaxation was also analysed. The relaxation of the tyre extends the unstable region, which can not be removed by damping only.

Investigate now the effect of the torsional stiffness and the torsional damping. The values of the two new dimensionless parameters, $K$ and $B$ represent the rates of the torsional and lateral stiffness and damping parameters in the system. The D–subdivision method combined with numerical algorithms lead to the stability charts in the parameter plane of the dimensionless towing speed $V$ and dimensionless caster length $L$. The stable domains are shaded in Fig. 2 for different value of $K$.

It can be observed, that the intersection of the stability boundaries, where quasi-periodic vibration was detected in [8], disappears already for small values of the dimensionless torsional stiffness $K$. For small towing speeds, the stationary rolling of the towed wheel becomes stable at zero caster length, if torsional stiffness is applied at the king pin. For nonzero values of the specific dimensionless torsional stiffness $K$, the instability domains, including the most relevant one, shrink.

In Fig. 3, the stability charts are plotted for a large $K$ while the dimensionless torsional damping $B$ is increased. As it is shown, the most relevant instability domain becomes an isolated island (or lens) for a critical dimensionless torsional damping, and the stationary rolling of the towed wheel turns to be stable for zero caster length for any towing speed.

## 6 Conclusions

The characteristic equation of an extended shimmy model was calculated and the stability charts were plotted for different parameter ranges of the system. As it is shown in Figs. 2 and 3, the stability behaviour of the towed wheel depends strongly on the torsional support at the king pin. It was shown that the stationary rolling at zero caster length is stable for any values of the towing speed, if both torsional spring and shimmy damper are applied at the king pin of the system. The straight-line rolling of the towed wheel can be stable for any values of the caster length and towing speed if the damping ratio of the system is large enough. For example, in the bottom right stability chart of Fig. 3, all the unstable domains disappear for the damping ratio $\zeta \geq 0.016$.

## References


1 **Stépán G**, *Chaotic motion of wheels*, Vehicle System Dynamics **20** (1991), no. 6, 341-351.

2 **Takács D, Stépán G, Hogan S J**, *Isolated large amplitude periodic motions of towed rigid wheel*, Nonlinear Dynamics ( 2007). appeared online.

3 **Pacejka H. B.**, *Tyre and Vehicle Dynamics*, Elsevier-Butterworth-Heinemann, Oxford, 2002.

4 **Sharp R S, Evangelou S, Limebeer D J N**, *Advances in the modelling of motorcycle dynamics*, Multibody System Dynamics **12** (2004), no. 3, 251-283.

5 **Takács D, Orosz G, Stépán G**, *Delay effects in shimmy dynamics of wheels with stretched-string like tyres*, European Journal of Mechanics (2007). Solid/A, to submitted.

6 **Segel L**, *Force and moment response of pneumatic tires to lateral motion inputs*, Transactions of the ASME **88B** (1966), no. 1, 37-44.

7 **Stépán G**, *Delay, nonlinear oscillations and shimmying wheels*, Applications of Nonlinear and Chaotic Dynamics in Mechanics (1998), 373-386.

8 **Takács D, Stépán G**, *Experiments on quasi-periodic wheel shimmy*, Proceedings of IDETC/CIE 2007 (2007). ASME.